

Measurement of high order current correlators.

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The feasibility of measuring high-order current correlators by means of a linear detector is analyzed. Two different types of measurements are considered: measurement of fluctuation power spectrum and measurement of unequal-time current correlators at fixed points in time. In both cases, formally exact expressions in terms of Keldysh time-ordered electron current operators are derived for the detector output. An explicit time ordering is found for the current correlators under the expectation operator used in measurements of high order unequal-time current correlators. The situation when a detector measures current correlators at different points of a conductor is considered.

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I. INTRODUCTION

Study of noise has recently been gaining popularity. This interest was quintessentially expressed by R. Landauer in the title of his paper *The Noise Is the Signal*[1]. It turned out that measurement of noise (electron current fluctuations) can be used to determine characteristics of electron transport that cannot be obtained by measuring only the average current or the conductance of the system.

The phenomenon of shot noise was predicted by W. Schottky as early as in the beginning of the past century [2]. Schottky showed that the variance of the fluctuating current carried by statistically independent discrete charges is proportional to the product of the carrier charge with the average current. This fact can be used to measure the carrier charge in various systems. Originally, shot noise was measured in electron vacuum tubes [3]. Much later, measurements of shot noise were used to determine the quasiparticle charge in edge states for quantum Hall systems [4, 5] and the double electron charge for Cooper pairs in hybrid NS systems [6]. Furthermore, shot noise measurement can be used to study the electron correlations due to fermionic statistics and electron-electron interaction effects (e.g. see review [7]).

Considerable interest in study of current correlators was also motivated by prospects of preparation of bipartite entangled states in superconductors [8] and normal electronic conductors [9], as well as of entanglement in quantum Hall effect [10]. The corresponding bipartite Bell inequality (which characterizes the degree of entanglement) is formulated in terms of second-order current correlators. Moreover, measurement itself can be a source of entangled electron pairs [9]. In [11, 12], a scheme was proposed for preparing an arbitrary n -electron entangled state, which requires measurement of n th-order current correlators.

Furthermore, measurement of second and higher order current correlators can provide as much information as possible about the state of a conductor under constraints imposed by quantum theory. This idea can be implemented in study of the full counting statistics of transmitted charge. Quantities of this kind have long since been analyzed in quantum optics in studies of the counting statistics of photons emitted by various sources (e.g. see [13]). However, a similar idea regarding electron systems was proposed only in relatively recent studies [14, 15], where the

distribution function was found for the charge transmitted through a quantum point contact over a finite time interval.

The past decade has seen a rapid increase in the number of theoretical studies focused on high-order current correlators, in particular in diffusive [16, 17], chaotic [18], and interacting [19, 20, 21] electron systems. Considerable attention was also given to the effects of temperature [22] and electromagnetic environment [23] on the third-order current correlator. A general approach to full counting statistics of current fluctuations, based on the path integral method, was recently developed in [20, 24].

Despite numerous theoretical studies, only zero- or finite-frequency shot noise has so far been amenable to direct measurement (see [4, 5, 6, 25, 26]). In [27], electron antibunching was studied in a fermionic Hanbury-Brown-Twiss-type experiment by measuring electron current cross correlations in a multiterminal conductor. Only in recent experimental studies, the third cumulant of electron current fluctuations was measured at low frequencies in a tunnel junction [28, 29] and in a quantum dot in the Coulomb blockade regime [30].

These measurements of the third cumulant stimulated theoretical studies where various quantum detectors of current fluctuations were proposed. They can be divided in two types: (1) detectors where transitions between discrete energy levels are induced by current fluctuations [31, 32, 33, 34, 35]; and (2) threshold detectors where transition from a metastable state is caused by interaction with current fluctuations [36, 37].

Since current operators taken at different times do not commute, practical measurement of current noise leads to an additional question about which particular correlation function is measured in an actual experiment. For second-order current correlators, the standard prescription is to use the symmetrized correlator $\langle \hat{I}(t_2)\hat{I}(t_1) + \hat{I}(t_1)\hat{I}(t_2) \rangle$ [38]. Indeed, as shown in [39] the symmetrized correlators at different times can be observed. Note, that at finite detector temperature, the anti-symmetrized correlator $i\langle \hat{I}(t_2)\hat{I}(t_1) - \hat{I}(t_1)\hat{I}(t_2) \rangle$ also contributes to the detector output due to the back action of the detector on the measured current. An analysis of measurement of noise power spectrum by means of a resonant LC circuit coupled to the conductor was presented in [40] (see also [31, 41]). In contrast to the measurement of the time resolved correlators, it was found that the ground-state

(passive) detector can measure only the positive-frequency noise power spectrum

$$S(\omega) = \int \langle \hat{I}(t) \hat{I}(0) \rangle e^{-i\omega t} dt,$$

where the current correlator is not time ordered. If the detector is in an excited state, then the negative-frequency power spectrum also contributes to the result. This behavior is explained by the fact that the ground state detector can only absorb energy, whereas an excited-state detector can transfer energy to the conductor as well.

An analogous time-ordering problem for current operators under the expectation operator arises with regard to measurements of high-order current correlators. In the analyses of full current statistics presented in [14, 15], it was shown that the transmitted charge cumulants $\langle\langle \hat{Q}^n(t) \rangle\rangle$ with $n \geq 3$ depend on the current measurement method. In particular, when $\hat{Q}(t)$ is defined as the integral $\hat{Q}(t) = \int_0^t \hat{I}(t') dt'$ without time ordering, as in the naive approach developed in [14], the resulting statistics corresponds to a fractional value of transmitted charge. However, an analysis of a more realistic measurement scheme using an auxiliary spin-1/2 system [15], revealed that the charge counting statistics (far from the scattering point at $x \gg x_c$, where for energy independent scattering x_c is of the order of the Fermi wave length) is described by a binomial distribution corresponding to an integer value of the charge. In the latter scheme, the cumulants $\langle\langle \hat{Q}^n(t) \rangle\rangle$ are expressed in terms of Keldysh time-ordered electron current operators. This discrepancy between current statistics raised a question about which correlator can be measured in an actual experiment.

In this paper, we theoretically analyze the feasibility of determination of high-order current correlators by measuring both fluctuation power spectrum and current correlators at fixed points in time. The present analysis does not address the dynamics of electrons in the conductor and behavior of current fluctuations per se. We consider the joint evolution of the conductor and the detector and find expressions for the detector output in terms of correlation functions of current operators, which can be used to determine all measurable correlation functions of current fluctuations. Since the arbitrary correlation functions of general form are considered, the results obtained here apply to measurement of observables of any quantum system coupled

to a detector. However, the analysis below is focused on the current measurement. Following [39, 40] we consider harmonic oscillator as a model of the detector coupled to the conductor at one or several points. In the general case, measurement of this kind corresponds to current measurement with a detector dynamics are described by a linear equation of motion.

II. GENERAL SCHEMES FOR MEASURING CURRENT FLUCTUATIONS

In this section, we describe two different schemes, which measure the power spectrum of current fluctuations at a finite frequency and current correlators at fixed times, respectively. We tentatively treat the detector as an arbitrary quantum system coupled to the conductor, assuming that measurement of its coordinate \hat{x} provides information about the magnitude of the current carried by the conductor. In the sections that follow, the role of a detector is played by a resonant LC circuit or a single-electron ammeter. Since either device can be modeled by a harmonic oscillator, the problem technically reduces to analysis of a harmonic oscillator driven by an external field (e.g. see [43]), which provides formally exact expressions for the detector output.

A. Continuous Quantum Measurement

Suppose that the coupling between the detector and the conductor is adiabatically switched on at an instant t_0 in the past. At $t > t_0$, a measurement is performed on the detector observable \hat{x} . The outcome of a quantum measurement on \hat{x} can be described only statistically in terms of the distribution function of the value of x , or, equivalently, in terms of moments of the form $\langle \hat{x}^n \rangle$, $n \in \mathbb{Z}$.

This procedure is a typical example of continuous quantum measurement: the detector continuously interacts with the measured system, and the detector output is read out either at several points in time with limited accuracy or only once (see review in [44] and references therein). The outcome is a time-averaged value of an observable, with an averaging kernel determined by the detector dynamics. In

particular, if the detector is modeled by a harmonic oscillator, then a cumulant $\langle\langle\hat{x}^n\rangle\rangle$ of lowest order in the coupling strength is proportional to the Fourier transform of an n th-order unequal-time current correlator or, equivalently, to the fluctuation power spectrum at the oscillator frequency (see below).

The generating function of \hat{x} defined as:

$$\chi(\lambda) = \text{Tr}_{\text{det}} \left\{ \hat{\rho}(t) \exp(i\lambda\hat{x}(t)) \right\}, \quad (1)$$

where $\hat{\rho}(t)$ is the detector density matrix and the trace is taken with respect to detector degrees of freedom. Once $\chi(\lambda)$ is known, one can find $\langle\hat{x}^n\rangle$ by differentiating $\chi(\lambda)$ with respect to λ :

$$\langle\hat{x}^n\rangle = (-i)^n \lim_{\lambda \rightarrow 0} (\partial_\lambda^n \chi(\lambda)), \quad (2)$$

The generating function is determined by the detector density matrix $\hat{\rho}$, which is obtained by tracing out the degrees of freedom of the measured system from the total density matrix \hat{D} : $\hat{\rho}(t) = \text{Tr}_{\text{sys}} \{\hat{D}(t)\}$. The density matrix $\hat{D}(t)$ is determined by the unitary evolution of the entire system starting from some point in the past. Suppose that the joint density matrix of the detector and the conductor at the starting point t_0 is a direct product of the form $\hat{D}(t_0) = \hat{\rho}_{in} \hat{R}_{in}$, where the density matrix \hat{R}_{in} represents the initial state of the conductor. We introduce a Hamiltonian \hat{H}_{sys} to describe free dynamics of the conductor and a Hamiltonian $\hat{H}_{\text{det}} = \hat{H}_0 + \hat{H}_{\text{int}}$, to describe detector dynamics, where \hat{H}_0 is the free detector Hamiltonian and \hat{H}_{int} is the coupling Hamiltonian. In the interaction representation with respect to \hat{H}_{sys} , the total density matrix at $t > t_0$ has the form:

$$\hat{D}(t) = \hat{S}(t_0, t) \hat{\rho}_{in} \hat{R}_{in}(t) \hat{S}^\dagger(t_0, t), \quad (3)$$

where the operator $\hat{S}(t_0, t)$ describes the evolution of the detector under the Hamiltonian \hat{H}_{det} :

$$\hat{S}(t_0, t) = \mathcal{T} \exp \left(-\frac{i}{\hbar} \int_{t_0}^t \hat{H}_{\text{det}}(t') dt' \right), \quad (4)$$

with \mathcal{T} is the forward time ordering operator.

We change to the \hat{x} representation for the total density matrix and the system evolution operator:

$$\hat{D}(x, y, t) = \langle x | \hat{D}(t) | y \rangle,$$

$$\hat{S}(x_0, t_0; x, t) = \langle x | \hat{S}(t_0, t) | x_0 \rangle,$$

which remains operator acting on the state of the conductor. Now, the characteristic function $\chi(\lambda)$ can be represented as:

$$\chi(\lambda) = Tr_{\text{sys}} \left\{ \hat{R}_{in}(t) \int dx e^{i\lambda x} \rho_{in}(x_0, y_0) \hat{S}^\dagger(y_0, t_0; x, t) \hat{S}(x_0, t_0; x, t) \right\}. \quad (5)$$

For $\lambda = 0$, the expression on the right-hand side here is the influence functional [43], describing the back-action of the detector on conductor dynamics.

B. Subsequent Quantum Measurements

Now, consider another feasible measurement scheme, where the detector state is measured at subsequent time moments $t_1 < t_2 < \dots < t_n$. According to von Neumann's projection postulate [45], an ideal quantum measurement performed on \hat{x} at any t_i projects the detector state at t_i onto an eigenstate of the operator \hat{x} . This measurement is strong in the sense that it strongly changes the detector wavefunction at point t_i and therefore affects its further evolution and measurement outcomes at $t > t_i$.

First, consider measurements performed on \hat{x} at two subsequent moments t_1 and t_2 . The resulting correlation function $\langle x(t_1)x(t_2) \rangle$ can be expressed in terms of the joint probability $P(x_1, t_1; x_2, t_2)$ that the values of \hat{x} measured at t_1 and t_2 are x_1 and x_2 , respectively:

$$\langle x(t_1)x(t_2) \rangle = \int dx_1 dx_2 x_1 x_2 P(x_1, t_1; x_2, t_2). \quad (6)$$

In turn, the joint probability $P(x_1, t_1; x_2, t_2)$ is given by the product:

$$P(x_1, t_1; x_2, t_2) = P(x_2, t_2 | x_1, t_1) P(x_1, t_1),$$

where $P(x_2, t_2 | x_1, t_1)$ is probability that the value of \hat{x} is x_2 at t_2 conditioned on the outcome x_1 of the measurement at moment t_1 , and $P(x_1, t_1)$ is the probability to measure the outcome x_1 at moment t_1 .

The last two probabilities can be determined from principles of quantum mechanics. If $\hat{D}(t)$ is the joint density matrix of the detector and the conductor, then

$$P(x_1, t_1) = Tr_{\text{sys}} \{ \langle x_1 | \hat{D}(t_1) | x_1 \rangle \}.$$

The conditional probability can be found by invoking von Neumanns projection postulate: at time moment t_1 , the total density matrix undergoes an instantaneous change:

$$\hat{D}(t_1^{-0}) \rightarrow \hat{D}(t_1^{+0}) = \frac{|x_1\rangle\langle x_1|\hat{D}(t_1^{-0})|x_1\rangle\langle x_1|}{Tr_{\text{sys}}\{|x_1\rangle\langle x_1|\hat{D}(t_1^{-0})|x_1\rangle\langle x_1|\}}, \quad (7)$$

where the denominator is introduced to normalize the reduced density matrix. At $t > t_1$, the reduced matrix is described by a unitary evolution operator:

$$\hat{D}(t_2) = \hat{S}(t_1, t_2)\hat{D}(t_1^{+0})\hat{S}^\dagger(t_1, t_2).$$

Thus, the conditional probability $P(x_2, t_2|x_1, t_1)$ is expressed as follows:

$$P(x_2, t_2|x_1, t_1) = Tr_{\text{sys}}\left\{\hat{S}(x_1, t_1; x_2, t_2) \frac{\hat{D}(x_1, x_1, t_1)}{Tr_{\text{sys}}\{\hat{D}(x_1, x_1, t_1)\}} \hat{S}^\dagger(x_1, t_1; x_2, t_2)\right\}, \quad (8)$$

where $\hat{D}(x, y, t) = \langle x|\hat{D}(t)|y\rangle$ is the total density matrix in the basis of eigenfunctions of the observable \hat{x} .

Taking the product of the conditional probability on $P(x_1, t_1)$, we find that the normalization factor cancels out. The result is

$$P(x_1, t_1; x_2, t_2) = Tr_{\text{sys}}\{\hat{S}(x_1, t_1; x_2, t_2)\hat{D}(x_1, x_1, t_1)\hat{S}^\dagger(x_1, t_1; x_2, t_2)\}. \quad (9)$$

Since the correlation function $\langle x(t_1)x(t_2)\rangle$ is calculated by averaging over all possible outcomes at t_1 and t_2 , the measurement performed at t_1 can be interpreted as the instantaneous diagonalization of $\hat{D}(x, y, t)$ in the basis of the measured observable,

$$\hat{D}(x, y, t_1^{-0}) \rightarrow \hat{D}(x, x, t_1^{+0}), \quad (10)$$

which is an equivalent formulation of von Neumanns projection postulate.

In the general case of subsequent measurements, the correlation function of the detector outputs at times t_i can be calculated as:

$$\langle x(t_1)...x(t_n)\rangle = \int \prod_{i=1}^n x_i dx_i P(\{x_i, t_i\}), \quad (11)$$

where $P(\{x_i, t_i\})$ is the joint probability that the detector coordinates at t_1, \dots, t_n are x_1, \dots, x_n respectively. For a single measurement of the detector coordinate \hat{x} , the probability $P(x_1, t_1)$ is expressed in terms of the diagonal elements of the detector density matrix at time t_1 :

$$P(x_1, t_1) = Tr_{\text{sys}}\{\hat{D}(x_1, x_1, t_1)\}.$$

By analogy, the probability $P(\{x_i, t_i\})$ for subsequent measurements can be expressed in terms of diagonal elements of the density matrix

$$\begin{aligned} \hat{D}(\{x_i, x_i, t_i\}) &= \int dx_0 dy_0 \hat{S}(x_{n-1}, t_{n-1}; x_n, t_n) \dots \hat{S}(x_0, t_0; x_1, t_1) \rho_{in}(x_0, y_0) \hat{R}_{in} \\ &\times \hat{S}^\dagger(y_0, t_0; x_1, t_1) \dots \hat{S}^\dagger(x_{n-1}, t_{n-1}; x_n, t_n), \end{aligned} \quad (12)$$

where the values of \hat{x} at $t = t_i$ have to be equal to each other under forward and backward time evolution. Thus, the detector density matrix at the times of measurements, $t = t_i$, is diagonal in the basis of eigenstates of \hat{x} . Therefore, the required probability is

$$P(\{x_i, t_i\}) = Tr_{\text{sys}}\{\hat{D}(\{x_i, x_i, t_i\})\}. \quad (13)$$

Note that the correlation function defined in this manner must be real-valued, because each measurement outcome is a real number by virtue of von Neumanns projection postulate. This correlation function should be distinguished from the correlation function $\langle \hat{x}(t_1) \dots \hat{x}(t_n) \rangle$ of coordinate operators, which may not be real-valued in the general case.

In this approach, each measurement projects the state of the system onto an eigenstate of the measured observable, as in von Neumanns ideal measurement. However, it is not the only type of feasible time resolved measurements. For example, the theory of photon detection in quantum optics assumes that the photodetector is reset to the ground state after each count and then switched back to the standby mode [13]. In distinction to (7), this measurement of $x(t_i) = x_i$ formally corresponds to an instantaneous transformation of the density matrix bringing the system into the next state,

$$\hat{D}(t_i^-) \rightarrow \hat{D}(t_i^+) = \hat{\rho}_i \otimes \frac{\langle x_i | \hat{D}(t_i^-) | x_i \rangle}{Tr_{\text{sys}}\{\langle x_i | \hat{D}(t_i^-) | x_i \rangle\}}, \quad (14)$$

which corresponds to the direct product of the detector density matrix $\hat{\rho}_i$ with the conductor one. The density matrix $\hat{\rho}_i$ represents the detector state in standby mode at time t_i . The corresponding correlation function of \hat{x} is expressed as:

$$\begin{aligned} \langle x(t_1) \dots x(t_n) \rangle &= Tr_{\text{det}}\{\hat{x} \hat{S}(t_{n-1}, t_n) \hat{\rho}_{n-1} \times \dots \times Tr_{\text{det}}\{\hat{x} \hat{S}(t_1, t_2) \hat{\rho}_1 \\ &\times Tr_{\text{det}}\{\hat{x} \hat{S}(t_0, t_1) \hat{\rho}_{in} \hat{R}_{in} \hat{S}^\dagger(t_0, t_1)\} \hat{S}^\dagger(t_1, t_2)\} \times \dots \times \hat{S}^\dagger(t_{n-1}, t_n)\}. \end{aligned} \quad (15)$$

In a sense, this type of measurement is more natural than von Neumann's ideal measurement. Indeed, the procedure described here assumes that the detector itself is a quantum-mechanical system whose state must also be measured. The state of the detector can be measured by coupling it to some macroscopic device. The coupling must be sufficiently strong for the detector state to be determined over a time interval much shorter than the interval between subsequent measurements. The resulting detector state may be different from $|x_i\rangle$, for example, in the ground state. This occurs when a detector excited to a metastable state by interaction with the measured system loses energy to the macroscopic device.

III. MEASUREMENT OF NOISE POWER SPECTRUM

Consider a measurement of the power spectrum of current fluctuations using a resonant LC circuit inductively coupled to a quantum conductor as a detector. Suppose that the observable to be measured is the electric charge $\hat{q}(t_1)$ stored in the capacitor at a point in time t_1 . As a model of the LC circuit, consider a weakly damped harmonic oscillator driven by an external force $\hat{J}(t) = \sum_i \alpha_i \hat{I}_i(t)$. The total Hamiltonian is

$$\hat{H}_{\text{det}} = \frac{L\hat{j}^2}{2} + \frac{\hat{q}^2}{2C} + \hat{j} \sum_i \alpha_i \hat{I}_i(t), \quad (16)$$

where L and C denote the inductance and capacitance of the circuit, the operators \hat{j} and \hat{I}_i represent the currents through the LC circuit and through the conductor at coordinate x_i ; and α_i denotes the mutual inductance between the LC circuit and the conductor in the neighborhood of x_i . The operators \hat{q} and \hat{j} can be expressed in terms of bosonic creation and annihilation operators:

$$\hat{q} = \left(\frac{\hbar}{2\Omega L}\right)^{1/2}(\hat{a}^\dagger + \hat{a}), \quad \hat{j} = i\left(\frac{\hbar\Omega}{2L}\right)^{1/2}(\hat{a}^\dagger - \hat{a}), \quad (17)$$

where $\Omega = 1/\sqrt{LC}$ is the resonant frequency of the LC circuit.

For the harmonic oscillator with Hamiltonian (16), the evolution operator in the \hat{q} representation can be found in explicit form:

$$\hat{S}(q_1, q_2, T) = S_0(q_1, q_2, T) \mathcal{T} \exp \left(\frac{iL\Omega}{\hbar \sin \Omega T} \left\{ \frac{q_1}{L} \int_{t_1}^{t_2} \hat{J}(t) \cos \Omega(t_2 - t) dt \right. \right. \quad (18)$$

$$-\frac{q_2}{L} \int_{t_1}^{t_2} \hat{J}(t) \cos \Omega(t - t_1) dt + \frac{1}{L^2} \int_{t_1}^{t_2} dt \int_{t_1}^t ds \hat{J}(t) \hat{J}(s) \cos \Omega(t_2 - t) \cos \Omega(s - t_1) \Big\} \Bigg),$$

where $T = t_2 - t_1$, and the transition amplitude for a free harmonic oscillator is expressed as

$$S_0(q_1, q_2, T) = \left(\frac{L\Omega}{2\pi i \hbar \sin \Omega T} \right)^{1/2} \exp \left(\frac{iL\Omega}{2\hbar \sin \Omega T} ((q_1^2 + q_2^2) \cos \Omega T - 2q_1 q_2) \right). \quad (19)$$

Assuming that the LC circuit prior to measurement (at $t_0 \rightarrow -\infty$) is in equilibrium at temperature Θ and performing the integral in (5) over the initial and final detector coordinates, we find that $\chi(\lambda)$ is the product of the equilibrium characteristic function $\chi_0(\lambda)$ with the characteristic function of excess capacitor-charge fluctuations:

$$\chi(\lambda) = \chi_0(\lambda) \left\langle \mathcal{T}_{\pm} \exp \left[\frac{i\lambda}{\hbar} \int_{t_0}^{t_1} (G(t_1 - t) \hat{J}_+(t) + G^*(t_1 - t) \hat{J}_-(t)) dt \right] F[\hat{J}_+, \hat{J}_-] \right\rangle, \quad (20)$$

where the influence functional of the detector $F[\hat{J}_+, \hat{J}_-]$ is expressed as

$$F[\hat{J}_+, \hat{J}_-] = \exp \left[-\frac{1}{\hbar^2} \int_{t_0}^{t_1} dt \int_{t_0}^t ds (\hat{J}_+(t) - \hat{J}_-(t)) (\Gamma(t - s) \hat{J}_+(s) - \Gamma^*(t - s) \hat{J}_-(s)) \right], \quad (21)$$

the time-ordering operator \mathcal{T}_{\pm} is introduced to order the current operators $\hat{J}_{\pm}(t)$ forward or backwards in time, and averaging is performed over the detector states. The response functions in (20) are defined as follows:

$$G(t) = \left(\frac{\hbar}{2L} \right) ((N + 1) e^{-i\Omega t} - N e^{i\Omega t}) \quad (22)$$

$$\Gamma(t) = \left(\frac{\hbar\Omega}{2L} \right) ((N + 1) e^{-i\Omega t} + N e^{i\Omega t}), \quad (23)$$

where $N = 1/(e^{\hbar\Omega/\Theta} - 1)$ is the bosonic occupation number for the oscillator mode. The equilibrium characteristic function $\chi_0(\lambda)$ corresponds to Gaussian charge fluctuations with variance $\delta q^2 = (\hbar/L\Omega)(N + \frac{1}{2})$:

$$\chi_0(\lambda) = \exp \left(-\frac{1}{2} \lambda^2 \delta q^2 \right). \quad (24)$$

Expressions (20) and (21) can be rewritten in more compact form in terms of the Keldysh Greens functions

$$G_K(t - t') = -i \langle \mathcal{T}_K \hat{q}(t) \hat{j}(t') \rangle,$$

$$\Gamma_K(t - t') = \langle \mathcal{T}_K \hat{j}(t) \hat{j}(t') \rangle,$$

where the operator \mathcal{T}_K denotes time-ordering along the Keldysh contour consisting of forward and backward branches (t_0, t_1) and (t_1, t_0) :

$$\begin{aligned} \chi(\lambda) = \chi_0(\lambda) & \left\langle \mathcal{T}_K \exp \left[\frac{i\lambda}{\hbar} \int_K G_K(t_1 - t) \hat{J}(t) dt \right] \right. \\ & \left. \times \exp \left[-\frac{1}{2\hbar^2} \int \int_K \Gamma_K(t - s) \hat{J}(t) \hat{J}(s) dt ds \right] \right\rangle. \end{aligned} \quad (25)$$

This expression provides a formally exact description of capacitor charge statistics for the LC circuit. The characteristic function $\chi(\lambda)$ can hardly be calculated in explicit form, but expression (25) makes it possible to develop a perturbation theory in the parameter $\alpha/L \ll 1$ by representing the influence functional $F[\hat{J}_+, \hat{J}_-]$ as a Taylor series expansion.

The analysis that follows focuses on the limit case when the back-action of the detector on the measured system, described by the influence functional given by (21), is negligible. Setting $F[\hat{J}_+, \hat{J}_-] = 1$, we obtain:

$$\chi(\lambda) = \chi_0(\lambda) \left\langle \mathcal{T}_K \exp \left[\frac{i\lambda}{\hbar} \int_K G_K(t_1 - t) \hat{J}(t) dt \right] \right\rangle. \quad (26)$$

First, we examine the autocovariance of the capacitor charge fluctuations. Using (2), we obtain:

$$\langle \hat{q}^2 \rangle = \delta q^2 + \frac{1}{\hbar^2} \left\langle \mathcal{T}_K \left(\int_K G_K(t_1 - t) \hat{J}(t) dt \right)^2 \right\rangle, \quad (27)$$

where the first and second terms correspond to equilibrium and excess fluctuations, respectively.

It should be noted here that perturbation series expansions contain secular terms [46]. Indeed, even though the exact expression for $\chi(\lambda)$ given by eq. (25) formally yields a finite result for any λ , each term in the perturbation series generally diverges as $t_1 \rightarrow \infty$. This is obviously true even for second-order quantities: in the stationary case, the current-current correlator in (27) is determined only by a time difference, and the resulting integral diverges as one of the remaining time variables goes to infinity.

These divergences can be eliminated by using weakly damped Greens functions. In physical terms, this means that a real detector not only interacts with the conductor, but also loses energy to the environment.

However, the introduction of an external heat bath is not a physical necessity. Indeed, its role can be played by the detector itself. The process can qualitatively be described as follows. An interaction with the conductor brings the detector into a steady state that can be strongly overheated as compared to its initial state. Any further excited state of the detector has a finite lifetime, because the detector will tend to lose energy to the conductor.

The degree of detector overheating can be estimated perturbatively. For the detector starting from the state with occupation number N , we use perturbation theory to calculate the probabilities of excitation to the state $N+1$ and de-excitation to the state $N-1$ over a time interval $\Delta t \gg \Omega^{-1}$:

$$P_{N+1}(\Delta t) = \frac{\alpha^2 \Omega \Delta t}{2\hbar L} (N+1) S^{(2)}(\Omega), \quad (28)$$

$$P_{N-1}(\Delta t) = \frac{\alpha^2 \Omega \Delta t}{2\hbar L} N S^{(2)}(-\Omega). \quad (29)$$

In perturbation theory, these probabilities are low. The power spectrum of current fluctuations is defined as

$$S_{ij}^{(2)}(\omega) = \int \langle \hat{I}_i(t_1) \hat{I}_j(t_2) \rangle e^{-i\omega(t_1-t_2)} d(t_1 - t_2). \quad (30)$$

Using the balance condition $P_{N-1} \geq P_{N+1}$, we estimate the steady-state occupation number as

$$\bar{N} \sim \frac{S^{(2)}(\Omega)}{S^{(2)}(-\Omega) - S^{(2)}(\Omega)}. \quad (31)$$

This quantity may be large even in the case of weak coupling between the detector and the conductor, which becomes obvious as the excitation energy tends to zero ($\Omega \rightarrow 0$).

However, we assume below that the detector is coupled to the environment, introducing the factor $e^{-\eta|t|}$ into integrals in all expressions analogous to (27). To estimate the leading-order contribution to the excess capacitor-charge variance $\langle \hat{q}^2 \rangle$, we express the average over detector states in (27) in terms of integrals of the noise power spectrum:

$$\langle \hat{q}^2 \rangle = \sum_{ij} \frac{\alpha_i \alpha_j}{(2L)^2} \left\{ (N+1) \int \frac{d\omega}{\pi} \frac{S_{ij}^{(2)}(\omega)}{(\omega - \Omega)^2 + \eta^2} - N \int \frac{d\omega}{\pi} \frac{S_{ij}^{(2)}(\omega)}{(\omega + \Omega)^2 + \eta^2} \right\}. \quad (32)$$

In the weak-damping limit ($\eta \ll \Omega$), we have $\eta/(x^2 + \eta^2) \rightarrow \pi\delta(x)$, and the result is expressed in terms of the positive- and negative-frequency power spectra, $S_{ij}^{(2)}(\Omega)$ and $S_{ij}^{(2)}(-\Omega)$:

$$\langle \hat{q}^2 \rangle = \sum_{ij} \frac{\alpha_i \alpha_j}{(2L)^2} \frac{1}{\eta} \left\{ S_{ij}^{(2)}(\Omega) + N(S_{ij}^{(2)}(\Omega) - S_{ij}^{(2)}(-\Omega)) \right\}. \quad (33)$$

Expression (33) extends the results presented in [40] to the case when the detector is coupled to the conductor at several points. The result for current measurement at two different points was presented in [47] and calculated later perturbatively in [48].

It is clear that negative-frequency current fluctuations contribute to this result only at finite temperatures ($N > 1$), whereas positive-frequency fluctuations contribute even at zero detector temperature. As noted in [40, 41] the reason is that energy transfer at zero temperature is possible only from the conductor to the detector, and the intensities of such processes are proportional to $S_{ij}^{(2)}(\Omega)$. At a finite temperature, the detector can be in an excited state from the start and transfer energy to the conductor at a rate proportional to $S_{ij}^{(2)}(-\Omega)$.

The proposed method can be used to measure any electron current correlation function. However, fluctuations of capacitor charge are dominated by fourth-order correlators. Indeed, it can easily be shown that the lowest order contribution to the mean stored charge,

$$\langle \hat{q} \rangle = \frac{1}{\eta} \sum_i \frac{\alpha_i}{L} \left(\frac{\eta}{\Omega} \right)^2 \langle \hat{I}_i \rangle, \quad (34)$$

is small since $\eta/\Omega \ll 1$. An analogous result holds true for any contribution to charge fluctuations containing odd-order correlators.

Now, consider measurement of fourth-order correlators. Retaining only the lowest order nonvanishing contribution to the fourth-order cumulant $\langle \langle \hat{q}^4 \rangle \rangle = \langle \hat{q}^4 \rangle - 3\langle \hat{q}^2 \rangle^2$, and using characteristic function (26), we obtain:

$$\langle \langle \hat{q}^4 \rangle \rangle = \frac{1}{\hbar^4} \left\langle \mathcal{T}_K \left(\int_K G_K(t_1 - t) \hat{J}(t) dt \right)^4 \right\rangle - 3\langle \hat{q}^2 \rangle^2. \quad (35)$$

For simplicity, suppose that $N = 0$ at the starting point. Then, the dominant contribution to capacitor charge fluctuations corresponds to the following time ordered average of current operators:

$$\langle \langle \hat{q}^4 \rangle \rangle = \frac{6}{(2L)^4} \int_{-\infty}^0 ds_1 ds_2 dt_1 dt_2 e^{i\Omega(t_1 + t_2 - s_1 - s_2)} \left\langle \mathcal{T}_- \{ \hat{J}(s_1) \hat{J}(s_2) \} \mathcal{T}_+ \{ \hat{J}(t_1) \hat{J}(t_2) \} \right\rangle. \quad (36)$$

The fourth-order unequal-time current correlator contained in this expression can be represented as the sum of an irreducible correlator and a number of reducible ones:

$$\langle \hat{I}_1 \hat{I}_2 \hat{I}_3 \hat{I}_4 \rangle = \langle \langle \hat{I}_1 \hat{I}_2 \hat{I}_3 \hat{I}_4 \rangle \rangle + \langle \hat{I}_1 \hat{I}_2 \rangle \langle \hat{I}_3 \hat{I}_4 \rangle + \langle \hat{I}_1 \hat{I}_4 \rangle \langle \hat{I}_2 \hat{I}_3 \rangle + \langle \hat{I}_1 \hat{I}_3 \rangle \langle \hat{I}_2 \hat{I}_4 \rangle + \dots,$$

where all odd-order correlators are dropped. Since steady-state current correlators depend only on time differences, the spectral density of the fourth-order irreducible correlator can be represented as

$$S_{ijkl}^{(4)}(\omega_1, \omega_2, \omega_3) = \int \prod_{i=1}^3 d(t_i - t_{i+1}) e^{-i\omega_i(t_i - t_{i+1})} \langle \langle \hat{I}_i(t_1) \hat{I}_j(t_2) \hat{I}_k(t_3) \hat{I}_l(t_4) \rangle \rangle, \quad (37)$$

whereas the reducible ones can be expressed in terms of the noise spectral density $S_{ij}^{(2)}(\omega)$ (cf. eq. (33)). Introducing a damping factor and performing the time integration, we obtain:

$$\begin{aligned} \langle \langle \hat{q}^4 \rangle \rangle &= 24 \sum_{ijkl} \frac{\alpha_i \alpha_j \alpha_k \alpha_l}{(2L)^4} \times \\ &\times \left\{ \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3} \frac{S_{ijkl}^{(4)}(\omega_1, \omega_2, \omega_3)}{(\omega_1 - \Omega - i\eta)((\omega_2 - 2\Omega)^2 + 4\eta^2)(\omega_3 - \Omega + i\eta)} \right. \\ &+ \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \frac{S_{ij}^{(2)}(\omega_1) S_{kl}^{(2)}(\omega_2)}{(\omega_1 + \omega_2 - 2\Omega)^2 + 4\eta^2} \left(\frac{1}{(\omega_2 - \Omega)^2 + \eta^2} \right. \\ &\left. \left. + \frac{1}{(\omega_1 - \Omega + i\eta)(\omega_2 - \Omega - i\eta)} \right) \right\} - 3 \langle \hat{q}^2 \rangle^2. \end{aligned} \quad (38)$$

Finally, we note that the output of a measurement of the power spectrum of current fluctuations performed with the same LC circuit at several points contains not only cross-covariances ($S_{ij}(\pm\Omega)$ for $i \neq j$), but also auto covariances ($S_{ii}(\pm\Omega)$ at the same point). To measure only cross-covariances, one must use two independent LC circuits with equal resonant frequencies coupled to the conductor at only one point. The results presented above can be extended to the case of several detectors.

In particular, for two LC circuits coupled to the conductor at points x and y , respectively, it can be shown that the charge correlator $\langle \hat{q}_x(t_1) \hat{q}_y(t_1) \rangle$ calculated to second order of perturbation theory is proportional to the second-order cross-covariance of current fluctuations:

$$\langle \hat{q}_x \hat{q}_y \rangle = \frac{\alpha_x \alpha_y}{(2L)^2} \frac{1}{\eta} \left\{ (N+1)(S_{xy}^{(2)}(\Omega) + S_{yx}^{(2)}(\Omega)) - N(S_{xy}^{(2)}(-\Omega) + S_{yx}^{(2)}(-\Omega)) \right\}. \quad (39)$$

IV. MEASUREMENT OF UNEQUAL-TIME CURRENT FLUCTUATIONS

Consider an experiment where current correlators are to be measured at certain points in time. In a thought experiment of this kind, we can use single-electron ammeter, whose readout is proportional to the exact instantaneous value of current. As a classical model of the ammeter, we can consider a damped harmonic oscillator driven by an external force proportional to current:

$$\ddot{x}(t) + \eta\dot{x}(t) + \Omega^2 x(t) = \alpha I(t), \quad (40)$$

where $x(t)$ is the ammeter readout, Ω is the resonant frequency, η is the damping factor, and α is the coupling constant. The solution to this equation can be represented in terms of the response function $x(t) = \int \chi(t-t')I(t')dt'$:

$$\chi(t) = \int \frac{d\omega}{2\pi} \chi(\omega) e^{-i\omega t} = \int \frac{d\omega}{2\pi} \frac{-\alpha e^{-i\omega t}}{\omega^2 - \Omega^2 + i\eta\omega}. \quad (41)$$

Since the oscillator motion is described by a linear equation, we can use the classical dynamical susceptibility $\chi(\omega)$ of the damped oscillator to determine temperature Greens functions by invoking the fluctuation dissipation theorem [49]. Accordingly, we first express the correlation function $\langle x(t_1)...x(t_n) \rangle$ of detector outputs in terms of Greens functions for the free oscillator and then replace these functions with those for the damped oscillator.

Thus, we assume that the detector output is the ammeter readout $\hat{x}(t)$, and the ammeter is described by the Hamiltonian:

$$\hat{H}_{\text{det}} = \frac{\hat{p}^2}{2m} + \frac{m\Omega^2 \hat{x}^2}{2} - \alpha m \hat{x} \hat{I}(t), \quad (42)$$

where the mass m and the momentum operator \hat{p} play the roles of moment of inertia and angular momentum, respectively. Unlike LC circuit Hamiltonian (16) it is linear in the coordinate rather than momentum of the oscillator.

In the coordinate representation, the amplitude of the transition from the state (x_1, t_1) to the state (x_2, t_2) driven by the external current $\hat{I}(t)$ is expressed as follows [43]:

$$\hat{S}(x_1, t_1; x_2, t_2) = S_0(x_1, x_2, T) \mathcal{T} \exp \left(\frac{im\Omega}{\hbar \sin \Omega T} \left\{ \frac{\alpha x_1}{\Omega} \int_{t_1}^{t_2} \hat{I}(t) \sin \Omega(t_2 - t) dt \right. \right. \quad (43)$$

$$+ \frac{\alpha x_2}{\Omega} \int_{t_1}^{t_2} \hat{I}(t) \sin \Omega(t - t_1) dt - \frac{\alpha^2}{\Omega^2} \int_{t_1}^{t_2} dt \int_{t_1}^t ds \hat{I}(t) \hat{I}(s) \sin \Omega(t_2 - t) \sin \Omega(s - t_1) \Bigg\},$$

where $S_0(x_1, x_2, T)$ is the transition amplitude for a free oscillator given by (19).

As shown above (see eq. (12)), in the case of an ideal measurement of the coordinate $\hat{x}(t_i)$ at each point in time, the correlation function $\langle x(t_1) \dots x(t_n) \rangle$ is calculated as the average over the intermediate coordinates of evolution operators of the form:

$$\int x_i dx_i \hat{S}_{i,i+1}^\dagger \hat{S}_{i-1,i}^\dagger \dots \hat{S}_{i-1,i} \hat{S}_{i,i+1},$$

where $\hat{S}_{ij} = \hat{S}(x_i, t_i; x_j, t_j)$. However, it is clear from expression (43) for the evolution operator that such an average is not well defined. Indeed, as a function of x_i this product of evolution operators contains the factor:

$$\begin{aligned} \mathcal{T}_\pm \exp \Bigg(\frac{i\alpha m x_i}{\hbar} \Bigg\{ \int_{t_{i-1}}^{t_i} (\hat{I}_+(t) - \hat{I}_-(t)) \frac{\sin \Omega(t - t_{i-1})}{\sin \Omega(t_i - t_{i-1})} dt \\ + \int_{t_i}^{t_{i+1}} (\hat{I}_+(t) - \hat{I}_-(t)) \frac{\sin \Omega(t_{i+1} - t)}{\sin \Omega(t_{i+1} - t_i)} dt \Bigg\} \Bigg), \end{aligned} \quad (44)$$

where \mathcal{T}_\pm denotes the time-ordering operators introduced to order the current operators $\hat{I}_+(t)$ and $\hat{I}_-(t)$ forward and backwards in time, respectively. The current operator under the time-ordering symbol can be treated as ordinary functions of time. This demonstrates that a weighted average of this form of the coordinate x_i is a divergent function of $\hat{I}_\pm(t)$.

One possible explanation of this divergency is that the oscillator with exactly known coordinate x_i has infinite energy. Accordingly, the probability that the oscillator will have a certain coordinate x_{i+1} at a subsequent point in time t is independent of x_{i+1}

$$P(x_{i+1}|x_i) = \frac{m\Omega}{2\pi\hbar \sin \Omega t},$$

(e.g. see [43]).

This unphysical result can be avoided by several means. First, we can assume that the coordinate is measured with limited accuracy $|x - x_i| \sim \Delta$. Then, the postmeasurement state of the oscillator is described by a density matrix localized within the measurement error, such as $\rho_{x_i}(x, y) \sim \exp(-[(x - x_i)^2 + (y - x_i)^2]/4\Delta^2)$. Second, as noted above, the measurement maps the detector density matrix to a certain standby state.

Hereinafter, we consider the latter scenario and assume that the measurement on $\hat{x}(t_i)$ brings the detector into an equilibrium state at temperature Θ . Then, the correlation function of detector outputs is given by formal expression (15). Integrating the evolution operators over the initial and final coordinates, we obtain:

$$\langle x(t_1) \dots x(t_n) \rangle = \left\langle \mathcal{T}_K F_K[\hat{I}] \prod_{i=1}^n \left(\frac{i\alpha}{\hbar} \int_{K_{i-1}^i} C_K(t_i - t) \hat{I}(t) dt \right) \right\rangle \quad (45)$$

where the Keldysh Greens function is defined as: $C_K(t - s) = \langle \mathcal{T}_K \{ \hat{x}(t) \hat{x}(s) \} \rangle$; $K_i^j = [t_i + i0 \rightarrow t_j \rightarrow t_i - i0]$ - is the Keldysh contour. The influence functional $F_K[\hat{I}]$:

$$F_K[\hat{I}] = \exp \left[-\frac{\alpha^2}{2\hbar^2} \int_K \int_K C_K(t - s) \hat{I}(t) \hat{I}(s) dt ds \right], \quad (46)$$

is defined on the Keldysh contour $K_{-\infty}^{\infty}$.

The Keldysh Greens function $C_K(t)$ can be expressed in terms of the causal Greens function $C(t > 0) = \langle \hat{x}(t) \hat{x}(0) \rangle$, which can in turn be expressed in terms of the dynamical susceptibility given by $\chi(\omega)$ (41), by invoking the fluctuation-dissipation theorem:

$$C(t) = \frac{\hbar}{\pi} \int d\omega e^{-i\omega t} \frac{\text{Im}\chi(\omega)}{1 - e^{-\hbar\omega/\Theta}}. \quad (47)$$

In what follows, Greens function $C(t)$ is conveniently represented as the complex variable: $C(t) = S(t) + iA(t)$, with real and imaginary parts corresponding to the symmetrized and antisymmetrized coordinate autocorrelation functions: $S(t) = \frac{1}{2} \langle \hat{x}(t) \hat{x}(0) + \hat{x}(0) \hat{x}(t) \rangle$, and $A(t) = \frac{1}{2i} \langle \hat{x}(t) \hat{x}(0) - \hat{x}(0) \hat{x}(t) \rangle$. Using the relation $1/(1 - e^{-x}) = \frac{1}{2} + \frac{1}{2} \coth(x/2)$, we obtain:

$$S(t) = \frac{\hbar}{2\pi} \int d\omega \cos(\omega t) \text{Im}\chi(\omega) \coth\left(\frac{\hbar\omega}{2\Theta}\right), \quad (48)$$

$$A(t) = -\frac{\hbar}{2\pi} \int d\omega \sin(\omega t) \text{Im}\chi(\omega). \quad (49)$$

A. Measurement of Second-Order Current Correlators

Here, we derive an expression for the correlation function $\langle x(t_1)x(t_2) \rangle$ in a perturbation theory in the coupling constant $\alpha \ll 1$ thus neglecting the back-action of the detector on conductor dynamics (i.e., by setting $F_K[\hat{I}] = 1$). The perturbative

calculation of $\langle x(t_1)x(t_2) \rangle$ was first carried out in Ref. [39]. Rewriting Eq. (45) in explicitly time-ordered form, we find [39]

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle = \frac{4\alpha^2}{\hbar^2} \int_{t_1}^{t_2} dt'_2 \int_{t_0}^{t_1} dt'_1 \{ & A(t_2 - t'_2)A(t_1 - t'_1) \langle \hat{I}(t'_2) \hat{I}(t'_1) \rangle^S \\ & - iA(t_2 - t'_2)S(t_1 - t'_1) \langle \hat{I}(t'_2) \hat{I}(t'_1) \rangle^A \}, \end{aligned} \quad (50)$$

where $\langle \hat{I}(t_1) \hat{I}(t_2) \rangle^S$ and $\langle \hat{I}(t_1) \hat{I}(t_2) \rangle^A$ are the symmetrized and antisymmetrized current correlators,

$$\langle \hat{I}(t_1) \hat{I}(t_2) \rangle^S = \frac{1}{2} \langle \hat{I}(t_1) \hat{I}(t_2) + \hat{I}(t_2) \hat{I}(t_1) \rangle, \quad (51)$$

$$\langle \hat{I}(t_1) \hat{I}(t_2) \rangle^A = \frac{1}{2} \langle \hat{I}(t_1) \hat{I}(t_2) - \hat{I}(t_2) \hat{I}(t_1) \rangle. \quad (52)$$

Since we focus on the measurement of current correlators at fixed times, t_1 and t_2 , the functions $A(t)$ and $S(t)$ in Eq. (50) must decay over time intervals much shorter than the time scale τ_e , of the evolution of the electron system. Otherwise, the correlator $\langle x(t_1)x(t_2) \rangle$ would be proportional to time-averaged current correlators. Calculating the integral over frequency in Eq. (49), we find that $A(t)$ is a decaying oscillating function independent of temperature:

$$A(t) = -\frac{\hbar}{2\omega_\eta} \exp\left(-\frac{\eta|t|}{2}\right) \sin(\omega_\eta t), \quad (53)$$

where $\omega_\eta = \sqrt{\Omega^2 - \eta^2/4}$ - is the renormalized oscillator frequency.

The symmetrized correlation function $S(t)$ is the sum of two functions, $S(t) = S_1(t) + S_2(t)$, having different long-time behavior. The decay of $S_1(t)$ is completely determined by the oscillator parameters:

$$S_1(t) = \frac{\hbar}{4\omega_\eta} e^{-\eta t/2} \left\{ e^{i\omega_\eta t} \coth \frac{\hbar(\omega_\eta + i\frac{\eta}{2})}{2\Theta} + C.c. \right\}, \quad (54)$$

whereas the decay of the other function depends on the oscillator temperature:

$$S_2(t) = -2\eta\Theta \sum_{n=1}^{\infty} \frac{\nu_n e^{-\nu_n t}}{(\nu_n^2 + \Omega^2)^2 - \eta^2 \nu_n^2}, \quad (55)$$

where $\nu_n = 2\pi n\Theta/\hbar$ is a Matsubara frequency.

Since the oscillator frequency is renormalized with Ω held constant, the functions $A(t)$ and $S_1(t)$ cannot decay faster than $e^{-\Omega t}$ (at $\eta = 2\Omega$) as $t \rightarrow \infty$. In the

strong-damping limit ($\eta \gg \Omega$), the damping factor is smaller, and both $A(t)$ and $S_1(t)$ decay approximately as $\sim e^{-\Omega(\Omega/\eta)t}$. Therefore, measurements of unequal-time correlators should be performed with oscillator parameters chosen so that $\eta \sim 2\Omega \gg \max\{\tau_e^{-1}, |t_2 - t_1|^{-1}\}$.

Since the decay of $S_2(t)$ depends on the detector temperature, it is necessary that $\nu_1 \gg \tau_e^{-1}$, which is equivalent to the requirement that $\Theta \gg \max\{\hbar\tau_e^{-1}, \hbar|t_2 - t_1|^{-1}\}$. For a ballistic conductor, the time scale of current fluctuations is $\tau_e \sim \hbar/eV$, where eV is the voltage across the conductor; therefore, the condition $\Theta \gg \hbar\tau_e^{-1}$ corresponds to $\Theta \gg eV$. This means that weakly nonequilibrium current fluctuations will be measured when the detector and the conductor have comparable temperatures.

In this approximation, the current operators in (50) can be factored out from the integral, and we can set:

$$\int_{t_1}^{t_2} A(t_2 - t) \hat{I}(t) dt \approx -\frac{\hbar}{2\Omega^2} \hat{I}(t_2). \quad (56)$$

$$\int_{t_1}^{t_2} S(t_2 - t) \hat{I}(t) dt \approx \frac{\eta\Theta}{\Omega^4} \hat{I}(t_2). \quad (57)$$

As a result, we find that the quantity measured at high temperatures is the following combination of the symmetrized and antisymmetrized current correlators:

$$\langle x(t_1)x(t_2) \rangle = \frac{\alpha^2}{\Omega^4} \left\{ \langle \hat{I}(t_2)\hat{I}(t_1) \rangle^S + i \left(\frac{2\eta\Theta}{\hbar\Omega^2} \right) \langle \hat{I}(t_2)\hat{I}(t_1) \rangle^A \right\}. \quad (58)$$

At $\eta \sim 2\Omega$ the relative contribution of either correlator depends on the ratio $4\Theta/\hbar\Omega$, which can have any value in this regime.

The high temperature result (58) in the limit $\omega_\eta \ll \eta$ was obtained in Ref. [39]. Another interesting experiment is measurement of quantum current fluctuations in the low-temperature regime ($\Theta \ll \hbar\tau_e^{-1}$). In this regime, measurement of highly nonequilibrium excess fluctuations becomes particularly feasible. In contrast to the high-temperature regime, the long-time decay of $S(t)$ follows a power law,

$$S(t \gg \Omega^{-1}) = -\left(\frac{\hbar\eta}{\pi\Omega^4} \right) \frac{1}{t^2}. \quad (59)$$

Again, requiring that $\eta \sim \Omega \gg \max\{\tau_e^{-1}, |t_2 - t_1|^{-1}\}$, we can assume,

$$\int_{t_1}^{t_2} S(t_2 - t) \hat{I}(t) dt \approx \hat{I}(t_2) \int_0^{t_2 - t_1} dt S(t) \approx \frac{\eta}{\Omega^4} \left(\Theta + \frac{\hbar}{\pi(t_2 - t_1)} \right) \hat{I}(t_2), \quad (60)$$

where in the last equation we have used the relation

$$\int_0^\infty S(t)dt = \eta\Theta/\Omega^4, \quad (61)$$

which holds at any temperature. The resulting expression for $\langle x(t_1)x(t_2) \rangle$ at low temperature of the detector takes the form,

$$\langle x(t_1)x(t_2) \rangle = \frac{\alpha^2}{\Omega^4} \left\{ \langle \hat{I}(t_2)\hat{I}(t_1) \rangle^S + i \frac{2\eta}{\hbar\Omega^2} \left(\Theta + \frac{\hbar}{\pi(t_1 - t_0)} \right) \langle \hat{I}(t_2)\hat{I}(t_1) \rangle^A \right\} \quad (62)$$

Comparing the last expression with the high-temperature result (58) we conclude that in the low-temperature limit the contribution of the antisymmetrized current correlator to the $\langle x(t_1)x(t_2) \rangle$ is proportional to the effective temperature,

$$\Theta' = \Theta + \frac{\hbar}{\pi\Delta t}, \quad (63)$$

where Δt is the time separation between subsequent measurement times.

B. Measurement of Higher Order Correlators

Perturbative treatment of Eq. (45) makes it possible to express a coordinate correlation function of any order in terms of current correlators. First, consider measurement of third-order correlators. Time-ordering the current operators along the Keldysh contour, we obtain

$$\begin{aligned} \langle x(t_1)x(t_2)x(t_3) \rangle = & -\left(\frac{2\alpha}{\hbar}\right)^3 \int_{t_2}^{t_3} dt'_3 \int_{t_1}^{t_2} dt'_2 \int_{t_0}^{t_1} dt'_1 \times \\ & \times \left\{ A(t_3 - t'_3)A(t_2 - t'_2)A(t_1 - t'_1) \langle \hat{I}(t'_3)\hat{I}(t'_2)\hat{I}(t'_1) \rangle^S \right. \\ & - iA(t_3 - t'_3)A(t_2 - t'_2)S(t_1 - t'_1) \langle \hat{I}(t'_3)\hat{I}(t'_2)\hat{I}(t'_1) \rangle^{SA_1} \\ & - iA(t_3 - t'_3)S(t_2 - t'_2)A(t_1 - t'_1) \langle \hat{I}(t'_3)\hat{I}(t'_2)\hat{I}(t'_1) \rangle^{SA_2} \\ & \left. + A(t_3 - t'_3)S(t_2 - t'_2)S(t_1 - t'_1) \langle \hat{I}(t'_3)\hat{I}(t'_2)\hat{I}(t'_1) \rangle^A \right\}. \end{aligned} \quad (64)$$

It is clear that four different third-order current correlators are measured in this case. Denoting by $\{A, B\}$ and $[A, B]$ the anticommutator and commutator of operators, respectively, we represent two of the four correlators as averages of symmetrized and antisymmetrized combinations of current operators taken at different times,

$$\langle \hat{I}_3\hat{I}_2\hat{I}_1 \rangle^S = \frac{1}{4} \left\langle \{ \{ \hat{I}_3, \hat{I}_2 \}, \hat{I}_1 \} \right\rangle, \quad (65)$$

$$\langle \hat{I}_3\hat{I}_2\hat{I}_1 \rangle^A = \frac{1}{4} \left\langle [[\hat{I}_3, \hat{I}_2], \hat{I}_1] \right\rangle, \quad (66)$$

and the other two as averages of mixed combinations of symmetrized and antisymmetrized current operators,

$$\langle \hat{I}_3 \hat{I}_2 \hat{I}_1 \rangle^{SA_1} = \frac{1}{4} \langle [\{\hat{I}_3, \hat{I}_2\}, \hat{I}_1] \rangle. \quad (67)$$

$$\langle \hat{I}_3 \hat{I}_2 \hat{I}_1 \rangle^{SA_2} = \frac{1}{4} \langle [\{\hat{I}_3, \hat{I}_2\}, \hat{I}_1] \rangle. \quad (68)$$

Note that these four Hermitian combinations of current operators are not the only possible ones. For example, following a standard prescription, we might assume that the completely symmetrized combination is measurable:

$$\langle \hat{I}_1 \hat{I}_2 \hat{I}_3 \rangle^{S'} = \frac{1}{6} \langle \hat{I}_1 \{\hat{I}_2, \hat{I}_3\} + \hat{I}_2 \{\hat{I}_3, \hat{I}_1\} + \hat{I}_3 \{\hat{I}_2, \hat{I}_1\} \rangle. \quad (69)$$

However, the analysis above shows that this is not true. The formal reason is that the current operators used to describe detector dynamics are Keldysh time-ordered. Since $t_1 < t_2 < t_3$, the combinations $\langle \hat{I}(t_2) \hat{I}(t_1) \hat{I}(t_3) \rangle$ and $\langle \hat{I}(t_3) \hat{I}(t_1) \hat{I}(t_2) \rangle$, which are contained in (69) and not contained in (65), can not appear as a result of such time-ordering. Indeed, the operator $\hat{I}(t_1)$, to be measured first will be on the right or left of other current operators, depending on whether time- or antitime-ordering is performed, respectively. In other words, the absence of $\langle \hat{I}(t_2) \hat{I}(t_1) \hat{I}(t_3) \rangle$ and $\langle \hat{I}(t_3) \hat{I}(t_1) \hat{I}(t_2) \rangle$ in correlator (65) is dictated by causality: measurements performed at later points in time t_2 and t_3 cannot affect the outcome of the earlier measurement at t_1 .

This argumentation suggests that a linear detector can be used to measure only correlators of the form:

$$\langle \hat{x}(t_1) \dots \hat{x}(t_n) \rangle \sim \langle \{ \{ \hat{I}(t_n), \hat{I}(t_{n-1}) \}, \dots \}, \hat{I}(t_1) \} \rangle, \quad (70)$$

where $t_1 \leq \dots \leq t_n$, as well as the remaining $2^{n-1} - 1$ correlators obtained by replacing any number of anticommutators with commutators.

Note that, depending on the measurement regime, the contributions of some of these 2^{n-1} correlators to $\langle x(t_1) \dots x(t_n) \rangle$ may be equal. In particular, consider the measurement of a third-order correlator,

$$\begin{aligned} \langle x(t_1) x(t_2) x(t_3) \rangle = & \frac{\alpha^3}{\Omega^6} \left\{ \langle \hat{I}(t_3) \hat{I}(t_2) \hat{I}(t_1) \rangle^S + i \left(\frac{2\eta\Theta'_1}{\hbar\Omega^2} \right) \langle \hat{I}(t_3) \hat{I}(t_2) \hat{I}(t_1) \rangle^{SA_1} \right. \\ & \left. + i \left(\frac{2\eta\Theta'_2}{\hbar\Omega^2} \right) \langle \hat{I}(t_3) \hat{I}(t_2) \hat{I}(t_1) \rangle^{SA_2} + \left(\frac{2\eta\Theta'_2}{\hbar\Omega^2} \right) \left(\frac{2\eta\Theta'_1}{\hbar\Omega^2} \right) \langle \hat{I}(t_3) \hat{I}(t_2) \hat{I}(t_1) \rangle^A \right\}, \end{aligned} \quad (71)$$

where $\Theta'_1 = \Theta + \frac{\hbar}{\pi(t_1-t_0)}$ and $\Theta'_2 = \Theta + \frac{\hbar}{\pi(t_2-t_1)}$ are two effective temperatures. In the high-temperature regime one has $\Theta'_1 \approx \Theta'_2 \approx \Theta$ and hence,

$$\begin{aligned} \langle x(t_1)x(t_2)x(t_3) \rangle &= \frac{\alpha^3}{\Omega^6} \left\{ \langle \hat{I}(t_3)\hat{I}(t_2)\hat{I}(t_1) \rangle^S \right. \\ &\quad \left. + i \left(\frac{2\eta\Theta}{\hbar\Omega^2} \right) \langle \hat{I}(t_3)\hat{I}(t_2)\hat{I}(t_1) \rangle^{SA_3} + \left(\frac{2\eta\Theta}{\hbar\Omega^2} \right)^2 \langle \hat{I}(t_3)\hat{I}(t_2)\hat{I}(t_1) \rangle^A \right\}, \end{aligned} \quad (72)$$

where the correlator $\langle \hat{I}_3\hat{I}_2\hat{I}_1 \rangle^{SA_3}$ is a combination of $\langle \hat{I}_3\hat{I}_2\hat{I}_1 \rangle^{SA_1}$ and $\langle \hat{I}_3\hat{I}_2\hat{I}_1 \rangle^{SA_2}$ having the form:

$$\langle \hat{I}_3\hat{I}_2\hat{I}_1 \rangle^{SA_3} = \frac{1}{2} \left(\langle \hat{I}_3\hat{I}_2\hat{I}_1 \rangle - \langle \hat{I}_1\hat{I}_2\hat{I}_3 \rangle \right). \quad (73)$$

V. CONCLUSIONS

We have theoretically analyzed measurements of higher-order current correlators with a quantum detector coupled to the conductor. A damped harmonic oscillator with coordinate $\hat{x}(t)$ measured directly is considered as a model of a detector whose classical dynamics are described by an equation of motion linear in x .

Two essentially different types of measurement have been considered: measurement of the power spectrum of current fluctuations and measurement of unequal-time current correlators at several points in time.

In the former measurement, a weakly damped resonant LC circuit inductively coupled to the conductor is used as a detector. In measurements of this type, the detector interacts with the conductor over a considerable time interval, and the measured observable \hat{q} is the charge stored in the capacitor of the LC circuit. Measurement of moments $\langle \hat{q}^n \rangle$ of the stored charge provides information about n th-order power spectrum of current fluctuations. A complete statistical description of \hat{q} is given by Eq. (25) in terms of integrals of Keldysh time-ordered current operators.

This equation is used to develop a perturbation theory in the coupling strength. By neglecting the backaction of the detector on the measured system, expression (34) is obtained for the second-order irreducible charge correlator $\langle \langle \hat{q}^2 \rangle \rangle$, in the case when the detector is coupled to the conductor at several points. At zero detector temperature, $\langle \langle \hat{q}^2 \rangle \rangle$ is proportional to the positive frequency Fourier transform of the second order current correlator symmetrized with respect to coordinates. At a

finite detector temperature, negative-frequency current correlator symmetrized with respect to coordinates also contributes to the stored charge.

Measurements of unequal-time current correlators are analyzed in the strong-damping limit, when the damping factor comparable to the oscillator frequency. The measured quantity is the correlation function $\langle x(t_1)...x(t_n) \rangle$ of the oscillator coordinate at subsequent points in time. The oscillator is assumed to relax to equilibrium after each measurement. This type of measurement is different from von Neumanns projective measurements at subsequent time moments. It is found that the correlation function $\langle x(t_1)...x(t_n) \rangle$ may be difficult to determine because an electron in a sharply localized state has infinite energy.

In the general case, the outcome of time resolved measurements of the detector state is described by Eq. (45). Perturbation theory is used to derive expressions (50) and (64) for second- and third-order coordinate correlators, respectively. When second-order correlators are measured, the detector output combines contributions of current correlators symmetrized and antisymmetrized with respect to time. In the high-temperature regime ($\Theta \gg eV$) the contribution to $\langle x(t_1)x(t_2) \rangle$ due to the antisymmetrized current correlator is proportional to the detector temperature Θ , see Eq. (58). At low temperatures, ($\Theta \ll eV$) one has the same result for $\langle x(t_1)x(t_2) \rangle$ as in the high-temperature regime with an effective detector temperature $\Theta' = \Theta + \hbar/(\pi\Delta t)$, where Δt is a time separation between subsequent measurements, see Eq. (62).

When higher order coordinate correlators are measured, a larger number of differently time-ordered current correlators contribute to the measurement outcome. The current operators in such correlators are Keldysh time-ordered in accordance with the causality requirement: a future current measurement cannot affect the outcome of a past measurement.

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